Magnetic Resonance Imaging and Fourier Theory

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Introduction

Nuclear Magnetic Resonance Imaging (NMRI) or, more commonly, *Magnetic Resonance Imaging (MRI)*, is a technique frequently used for visualising the inside of the human body for medical research or treatment. It is based on *Nuclear Magnetic Resonance (NMR)*; the phenomenon that certain nuclei can absorb radiofrequency energy, if these nuclei are contained within an external magnetic field. This phenomenon is briefly described in the first chapter. NMR is a phenomenon not limited to organic materials; therefore, variants of MRI exist which are used for non-medical purposes. If the goal still is to obtain a visualisation of the interior of an object, then the underlying physics and mathematics which are discussed here remain useful.

This work primarily concerns with the link between MRI and Fourier theory and will assume the reader to be somewhat familiar with the basics of *Fourier Transforms (FT)*. We will work from a simplified physical description of the basics of the MRI technique towards a more mathematical description thereof and relate this to FT. Eventually, we will see that FT can be naturally applied to NMR. This is in contrast to a different technique used in medical visualisation; *Computed Tomography (CT)*. CT also commonly uses FT to reconstruct an image [**1**], but the FT is not applied in a 'pure' form; it is not applied to harmonic functions occurring in physics.

CHAPTER 1

The Physics

1. Magnetic Resonance

1.1. Magnetic Spin. NMR is based on the magnetic spin property of elements. We will only touch the surface of this subject here. *Atoms* consist of neutrons, protons and electrons. A *nucleus* of an atom is the collection of all its neutrons and protons. The collective spin of a nucleus is called the *nuclear spin*. This spin is determined by the spins of the neutrons and protons it is made up from. Neutrons, protons and electrons have a spin of $\frac{1}{2}$ or $-\frac{1}{2}$, where the sign indicates the direction of spin. This spin denotes the *angular momentum* of an element. This angular moment is often compared to the rotational movement of the earth's axis. This is pictured in figure 1. A particle with spin and an electrical charge, has a magnetic moment; as such, nuclei also posses magnetic moments (as long its spin is nonzero) [**3**, Ch. 3] [**4**].

1.2. Alignment to an external magnetic field. Particles posessing magnetic moments have bipolar magnetic properties, as figure 1 implies. Placing nuclei with nonzero spin in an external magnetic field thus aligns the spin vectors to a more or less parallel position to the magnetic field direction. This is pictured in figure 2(a). Note that in this figure, the spin vector is aligned in a way such that the magnetic south point of the spin vector points to the magnetic north of the external field, and vice versa. This alignment is called the *low energy state*. An opposite alignment as in figure 2(b) is called the *high energy state*[1].

1.3. Changing the spin alignment. When a nucleus with nonzero spin is placed in a magnetic field, its spin vector becomes aligned. A particle aligned in a low energy state can be forced into a high energy state if we supply that particle with enough energy. This energy can be supplied in the form of a photon. A spin vector will only topple into a high energy state if the photon frequency ν equals [**3**, Ch. 3]:

$$\nu = \gamma B \quad (1)$$

[1]When initially placed in a magnetic field, the number of particles with spin in a low energy state (N^-) with respect to the particles in high energy state (N^+) is given by $N^-/N^+ = e^{-E/kT}$. E denotes the energy difference between the two spin states, k is the Boltzmann constant and T is the temperature [**3**].

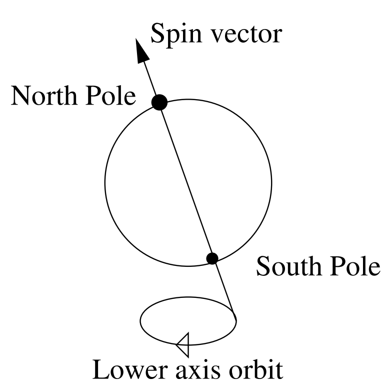

FIGURE 1. Abstract display of an sphere-shaped particle experiencing spin.

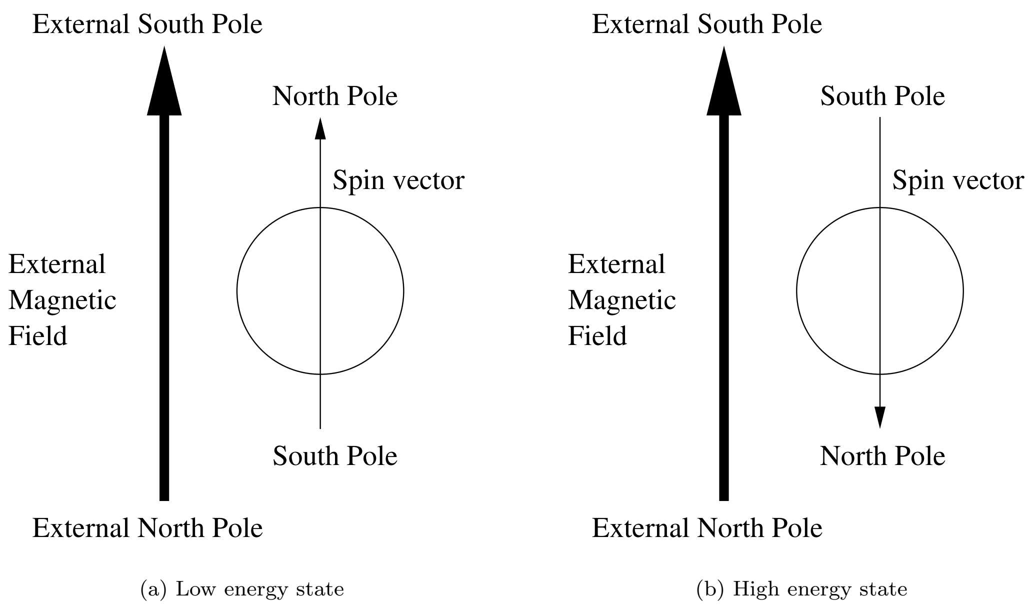

FIGURE 2. Spin vector alignment in an external magnetic field

where γ denotes the so called *gyromantic ratio* of the particle, and B the external magnetic field strength. ν is also called the *resonance frequency* or the *Larmor* frequency. Also, the energy of the photon must equal the energy difference between the low and high energy states of the particle [**3**, Ch. 3]. Note that the energy E and frequency ν of a photon are linked by

$$E = h\nu \tag{2}$$

where h is Planck's constant. (1) and (2) together yield that a photon causes a low to high energy transition when

$$E = h\gamma B \tag{3}$$

equals the difference in energy between the low and high energy states. In practise, required photon frequencies are within radio frequency (RF) ranges[2]. Hence, spin transition is achieved by sending RF signals. When a nuclei is brought into a high energy state, we say it is *excited*.

Globally, the number of high energy state nuclei tends to go to an equilibrium. When we artificially excite nuclei with RF pulses, we disturb this equilibrium. With time, after the RF pulse, some nuclei will flip back to their low energy states as the global tendency is to revert back to the equilibrium. This process is called relaxation.

If we consider the net magnetisation vector N of multiple nuclei, this vector will point along the magnetic field direction when in equilibrium. The *longitudinal* component of N is the component along the magnetic field direction, and the *transversal* components are those on a plane perpendicular to the magnetic field direction. See also figure 3.

If the nuclei we observe here are not in equilibrium, the transversal components may be nonzero and the longitudinal component will be below its maximum value. Upon relaxation, the longitudinal component will return to its maximum value N_0, and the transversal components will go to zero. The first process (longitudinal) is called the T1-process, and the latter process is called the T2-process.

[2]For MRI, ν is in between 15 and 80 MHz [**3**].

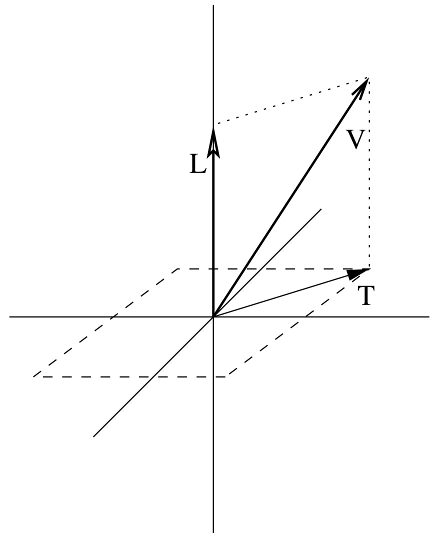

FIGURE 3. A vector V with is longitudinal component L and its transversal component T.

1.4. Spin readout. From basic physics, it is known that a non-static magnetic field induces an electrical current in a wire placed within it. If we place nuclei with magnetic moments within a constant external magnetic field, both the external field and the magnetic fields of the nuclei remain static; a wire placed anywhere in the magnetic fields will not experience any electrical current. However, if we send in an RF pulse at the right frequency, many particles aligned in the low energy state will flip over to a high energy state. This implies movement of the thereby belonging spin vectors; the magnetic field induced by the particles no longer is static and as such a well-placed wire will experience electrical current.

Suppose we place our particles within a Cartesian coordinate system with an x-, y- and z-axis. Also, let this system be oriented such that the external magnetic field is perfectly parallel to the z-axis. We can then place a coil[3] somewhere on the sides of the object we wish to scan on the xy-plane. By monitoring the current the coil is experiencing during and after an RF pulse, we obtain a signal [**2**]. This signal is what drives MRI and a series of these signals is fed directly to the Fast Fourier Transform (FFT) algorithm, as we will see later on.

2. MRI ingredients

Now suppose we have an object contained within a static magnetic field. We now know we can emit an RF pulse to excite nuclei with spin within that object, causing magnetic field variations which can be detected by means of a coil.

In most clinical applications the nuclei we want to excite is hydrogen (^{1}H)[4]. Hydrogen is primarily present in fat and tissues, but not in the same amount; some tissues resonate better since they posess more hydrogen then others, so to say. This difference in resonance is very useful of course; different tissues will look differently on MRI obtained pictures.

However, exciting *all* hydrogen nuclei in an object will not give us a useful signal.

The signal received by the coil during and after an RF pulse would not be solely caused by the nuclei on a single xy-plane. All nuclei on other locations along the z-axis would also contribute. To counter this we make use of the concept of *slice selection.*

2.1. Slice encoding. Remember firstly that the resonance frequency is dependent on the magnetic field strength (equation 1). Suppose we linearly variate the magnetic field strength in

[3]In NMR, the so called Barker coil is commonly used [**5**]

[4]^{1}H is the most common isotope of H in nature, by a factor of 99.9985%. It has a gyromantic ratio of 42.58 MHz per Tesla [**3**].

the direction of the z-axis. Then we may use RF pulses at different frequencies to excite only the nuclei on the xy-plane at specific z-coordinates. Particles on different locations on the z-axis will now not interfere with the signal of the nuclei at the selected xy-plane[5]. Therefore, our problem now is reduced to finding hydrogen nuclei at different locations in the xy-plane, using the fact that these nuclei are in an excited state.

Applying a linear variation to the external magnetic field strength is called a *field gradient*. We will denote this particular gradient by G_Z, since it is applied alongside the z-axis. Suppose we subdivide the object we want to scan into small cubic volumes called *packets*. These packets may contain different concentrations of hydrogen nuclei. If we add up the spin vectors of all nuclei in a single packet, we obtain a *net magnetisation vector*. Note now that when an RF pulse hits a packet, not all nuclei will flip immediately; the number of flips is determined by the strength and duration of the RF pulse. Hence, by controlling the RF strength and duration, we can let the net magnetisation vector make a specific angle θ between the z-axis and the xy-plane. $\theta = 0$ then denotes the original net magnetisation state; i.e., aligned to the z-axis, and $\theta = \pi$ denotes a downwards pointing net vector.

The RF pulses we apply with slice encoding, are made sure not to be strong enough (or not be applied long enough) to completely push the net magnetisation vector downwards (which would be the case if all nuclei were in the high energy state). Instead, we aim to push the spin vector into the xy-plane; that is, the spin vector is pushed 90 degrees ($\theta = \frac{\pi}{2}$) from its original position. This is called the 90 degrees RF pulse[6]. Note that since the net magnetisation vector now is in the xy-plane, the magnetic fields generated by the nuclei with spin are now detectable by the coil on the edges of the xy-plane.

The following equations describe the relaxation processes T1 and T2 after a 90 degrees pulse ending at $t = 0$. T_1 and T_2 are constants which depend on the matter which is being imaged.

$$M_{long} = M_0(1 - e^{-t/T_1}) \tag{4}$$

$$M_{tra} = M_0 e^{-t/T_2^*} \tag{5}$$

2.2. Frequency encoding. With frequency encoding, we apply a magnetic field gradient along the x axis. Denoting the gradient by G_X, we have that equation (1) gives that the resonance frequency of nuclei at a point x is equal to

$$n = \gamma(B_0 + xG_X) \tag{6}$$

where B_0 is the magnetic field strength at point $x = 0$. Hence, if we have packets with net spin in the xy-plane, applying the G_X field gradient causes the packets at different x locations to rotate *at different frequencies*. Concluding, frequency encoding can be used to differentiate between spins on various positions along the x-axis.

2.3. Phase encoding. Suppose we leave the G_X gradient in frequency encoding on for a little while. Then, since the spins rotate at different frequencies along the x-axis, they generally will not point at the same direction at a given time. Note however that if the gradient is turned off, that the Larmor frequencies are again equal along the x-axis. The phases however, differ; phase encoding makes use of this 'ending effect' of frequency encoding.

Assume we can also vary the magnetic field strength along the y-axis. We can then turn on a G_Y gradient and quickly turn it off again. Then the Larmor frequencies will be equal along the y-axis. But, since the spins were rotating at different speeds before, the phase of the spins along the y-axis will *not* be the same. This is called *phase encoding* and is used to differentiate between spins at different positions along the y-axis.

[5]Actually, the selected slice thickness is of course not equal to zero; also, the thickness of the slice is directly proportional to the magnetic gradient size.

[6]The duration and strength of a 90 degrees RF pulse are obtained by carefully calibrating prior to experiments. See for example [**7**].

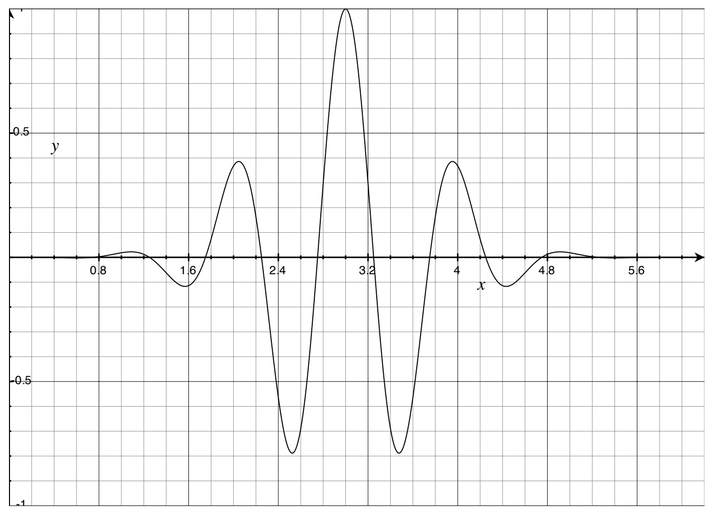

FIGURE 4. A sample MRI signal.

2.4. Putting it together. From the last three paragraphs we have seen ways to differentiate between nuclei on the z-axis (slice encoding), and the x- and y- axis (frequency / phase encoding)[7]. By applying these three techniques in a smart order, we obtain a basic way to obtain an MRI signal.

At first we start with applying a magnetic field gradient G_Z, together with a 90 degrees RF pulse. This results in detectable magnetic fields generated by the nuclei we want to image. After the net magnetic vectors are pushed in the xy-plane, the RF pulse and the G_Z gradient are turned off. Following, we turn on the G_X[8] gradient and turn it off after a specific amount of time has passed; this is the phase encoding. Finally, the G_Y gradient is turned on (frequency encoding) and a signal is recorded. This signal consists of all spins in the selected xy-plane all rotating at different phases and frequencies, dependent on their position. A singal is often pictured as in figure 2.4.

If we have a way to isolate the different phases, frequencies and amplitudes, we can construct a magnetic resonance image.

[7]Although frequency encoding was earlier explained while using the x-axis and the phase encoding using the y-axis, the choice which axis is which is completely arbitrary. This does not apply for slice encoding, since the z-axis is unambiguously defined as being aligned with the external magnetic field.

[8]Again, the gradient here could also have been the G_Y gradient.

CHAPTER 2

The Mathematics

We will firstly repeat some basic Fourier theory which is used in conjunction with MRI. For simplicity, we will assume the MRI signal we have obtained is a simple one obtained by recording a signal after a 90 degrees RF pulse; no 180 pulses will be applied[1]. This is so that the signal we have now can be described as a multiplicand of a harmonic function and an exponentially decreasing function. A spin-echo singal would be somewhat more difficult to describe, which would make some sections here needlessly complex.

1. Fourier transform

1.1. The time / frequency domains. The Fourier Transform (FT) of any function $f \in L^2$ (that is, functions f for which $\int_{-\infty}^{\infty} |f(t)|^2 dt < \infty$) is given by

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i \omega t} dt \tag{7}$$

Suppose we have a signal represented by the function f. If we assume f is T-periodic and $f \in L^2$; we can interpret f as the time domain signal and the FT $\hat{f}$ as the frequency domain function.

Time domain. $f(t)$ is said to be in time domain because this function represents the received signal; signals are of course obtained by measuring something over a specific time interval. Consider for example a coil placed next to a rotating magnet. The coil will experience a variating current depending on the alignment of the magnet with respect to the coil; in fact, if we plot the current v at a given time t we will obtain a sinusoid.

Frequency domain. Note the inverse FT, again for $f \in L^2$

$$f(t) = \int_{-\infty}^{\infty} \hat{f}(\omega) e^{2\pi i \omega t} d\omega \tag{8}$$

Let us consider for a moment the following function:

$$h(t, \omega) = \hat{f}(\omega) e^{2\pi i \omega t} = \hat{f}(\omega)(\cos(2\pi\omega t) + i\sin(2\pi\omega t)) \tag{9}$$

If we fix ω at some point, h becomes a function only dependant on the time t. This means that h is a purely harmonic function on the real and imaginary axes, respectively. Moreover, those individual harmonic functions ($\cos(2\pi\omega t)$ and $i\sin(2\pi\omega t)$) have exactly period ω. Following, we may conclude that the inverse FT is obtained as an combination of harmonic functions with periods $\omega \in (-\infty, \infty)$ and weights $\hat{f}(\omega)$. Thus if $\hat{f}(\omega)$ is nonzero solely on a fixed ω_1, f will be an ω_1-periodic function with real and imaginary components depending on $\hat{f}(\omega_1)$ [**1**, p. 31]. The amplitude of the frequency ω present in the signal f is equal to $|\hat{f}(\omega)|$.

[1]See the appendix about 180 degrees RF pulses and the spin-echo sequence.

1.2. Fourier transforms on harmonic functions with phase shifts. Remember the two basic rules for phase shifts of the sine and cosine functions:

$$\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y) \tag{10}$$

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y) \tag{11}$$

If we assume a constant phase shift $y \in \mathbb{R}$, then it is clear that a shifted harmonic function (sine or cosine) of a given frequency is equal to a summation of a sine and cosine functions of that frequency weighted by a factor dependent on the phase shift. When FT is applied to a phase shifted harmonic function, the weight $\hat{f}(\omega)$, with ω the frequency of the original signal, is exactly equal to the factors introduced in (10) and (11) if the original amplitude equals 1. Hence, if we consider $f(t) = c\cos(2\pi t + \phi)$, $c, \phi \in \mathbb{R}$:

$$\hat{f}(\omega) = \begin{cases} c(\cos(\phi) - i\sin(\phi)), & \text{if } |\omega| = 1 \\ 0, & \text{if } |\omega| \neq 1 \end{cases} \tag{12}$$

A similar relation can of course be derived for the case $c\sin(2\Pi t + \phi)$. We introduced here an amplitude c to show how that we still can recalculate the phase shift if the amplitude is not equal to 1. Note that $|c(\cos(\phi) - i\sin(\phi))| = \sqrt{c^2(\cos^2(\phi) + \sin^2(\phi))} = |c|$. Hence, when $\hat{f}(\omega) \neq 0$ and $|c| = c^2$, we have that $\frac{\hat{f}(\omega)}{|\hat{f}(\omega)|} = \cos(\phi) - i\sin(\phi) = e^{-i\phi}$. Concluding, we can use the following equality to calculate the phase shift ϕ:

$$\hat{f}(\omega) = |\hat{f}(\omega)|e^{-i\phi} \tag{13}$$

Note that from (13) it becomes evident that FT can not separate two signals of the same frequency with different phase; since the frequency is similar, there will be only one weight $\hat{f}(\omega)$ and thus we can only derive one phase shift. Indeed, if we have a sum of shifted harmonic functions, we can use (10) and (11) to rewrite the sum into only one shifted (co)sine.

1.3. Convolution. The convolution product of two functions f and h on $\mathbb{R}$ is defined as follows [**1**, p. 55]:

$$f * h(t) = \int_{-\infty}^{\infty} f(t-s)h(s)ds = \int_{-\infty}^{\infty} f(s)h(t-s)ds \quad \text{for } t \in \mathbb{R} \tag{14}$$

This stands in relation to the FT by the following theorem, see [**1**, p. 57].

THEOREM 1.1. *Let $f, h \in L^1(\mathbb{R}) \cup L^2(\mathbb{R})$. Also let $\hat{f}$ and $\hat{h}$ be the FT of f and h, respectively. Then:*

$$\widehat{(f * h)} = \hat{f}\hat{h} \tag{15}$$

2. Fourier theory and magnetic resonance

2.1. Magnetic spin readout revisited. We saw in the previous chapter that we receive a signal from all nuclei from a xy-slice after a 90 degrees RF pulse is applied. This pulse pushes the net magnetisation vectors of the nuclei on this slice into the xy-plane. These net vectors will rotate with a frequency dependant on the magnetic field strength, RF pulse duration and strength. The phase of the spins will depend on the phase encoding gradient. However, the vectors will not remain in the xy-plane as relaxation occurs. We have to take into account equation (5). The harmonic functions $h_{\phi,\xi}(t)$ represent the magnetic net vectors of the packets. These are dependant on given frequencies ϕ and phases ξ and are multiplied by the exponential function $M_{tra}(t)$ from (5). Their

[2] We could allow for c to be negative, however, the FT will always assume a positive amplitude and thus would shift the resulting function to the correct solution. Hence, for readability, we demand a positive amplitude.

sum yields approximately the final signal $f(t)$ we receive. Note that by the convolution theorem we may write the following:

$$\hat{f}(\omega) = \widehat{h_{\phi,\xi}} * \widehat{M} \tag{16}$$

Hence $\hat{f}(\omega)$ has a series of peaks around each frequency in ϕ. The amplitude of these peaks is dependant on the phases ξ, and the amplitude of h at the given positions ϕ and ξ.

2.2. Extracting phase information. Let us assume phase encoding took place over the y-axis. Then different spin packets at different locations along the y-axis have indeed a different phase. However, we have already seen that a normal FT can not distinguish between different phases. We therefore apply a different FT, namely a two dimensional FT. By [**1**, Ch. 4] we have that in such a case:

$$\hat{f}(\omega_1, \omega_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t_1, t_2) e^{-2\pi i(\omega_1 t_1 + \omega_2 t_2)} d\vec{t}, \quad \text{with } \vec{t} = (t_1, t_2) \tag{17}$$

If we take t_1 to be G_y, the phase encoding gradient, t_2 the elapsed time in which we receive the signal, and $f(t_1, t_2)$ the recorded signal we see something useful arise. Applying the integral (17) over t_2 will give us information about the frequencies contained in the signal. Applying over t_1 first gives something different. Note that $f(\xi_1, t) = f(\xi_2 - \Delta\xi, t)$ $(\Delta\xi = \xi_2 - \xi_1)$[3]; for different values t_1, $f(t_1, t_2)$ is a shifted version of $f(\xi_1, t_2)$.

If $f(t_1, t_2)$ was harmonic with t_1 fixed, a FT in the t_1 direction would then yield the *phase shift*. This becomes more apparent if we take for example $f = h(t_1, t_2) = cos(2\pi t_2 + t_1)$. With $\hat{h}(t_1, \omega_2)$ we denote the FT in the t_2-direction. We have already seen that at $\omega_2 \neq 1$ the function will be zero. At $\omega_2 = 1$ the function will yield a value dependent on t_1. If we on the other hand fix t_2 instead of t_1, we obtain the function $\tilde{h_{t_2}} = cos(2\pi t_2 + t_1)$, with the factor $2\pi t_2$ *constant*. Now $2\pi t_2$ has become the phase shift and $\tilde{h_{t_2}}$ has become a periodic function with its period dependant on t_1.

In our case, t_1 is dependent on the phase gradient and on the position of the packet along the y-axis, and on how long the phase encoding gradient was applied. Hence if we vary the phase gradient by a constant amount and record the signal belonging to each different gradient, we obtain linearly shifted harmonic functions which in the t_1 direction varies with a frequency related to those phase shifts.

Concluding, applying a complete 2D FT on any harmonic function will give us a function $I = \hat{f}(\omega_1, \omega_2)$ where ω_1 denotes a phase shift, and ω_2 a frequency. I is the intensity giving us information if the signal f contains an (co)sine with the given phase and frequency is present, and if it is, what its amplitude is. This is exactly what we need for MRI. Note however that the signal obtained in MRI is not purely harmonic, but that equation (16), provides us means to justify 2D FT in our case.

This is because that equation tells us we can split the MRI signal $f(t_1, t_2)$ into a harmonic part and into the relaxation part. Applying the 2D FT on the harmonic part gives us the data we actually want. If this is convoluted with the 2D FT of $M(t_1, t_2)$, we obtain a series of peaks around the frequencies and phases a signal was observed[4]. The height of these peaks is proportional to the intensity of the harmonics at the given phase and frequency. In the end, the peak heights are also proportional to TR, TE, T_1 and T_2 [5]:

$$I = c(1 - e^{-TR/T_1})e^{-TE/T_2}, \quad \text{taken from [4]}). \tag{18}$$

where c is a factor which may depend on variables other than T_1, T_2, TE and TR.

[3] In practise, this would of course not be precisely equal due to measurement errors, subtle environmental changes, etcetera.

[4] Note that M is solely dependent on the time elapsed after a 90 degrees RF pulse. Therefore it is independent of the phase shift t_1 and thus $M(t_1, t_2) = M(t_2)$.

[5] amongst other things, see [**4**].

Note that this way of handling the phase encoding requires us to have multiple signals at different gradient encodings. If we want to use k different phase encoding gradients, then we denote the ith gradient by $G_{y,i}$, $0 \leq i \leq k$. Usually, $G_{y,i} = 2i\frac{G_{max}}{k} - G_{max} \in [-G_{max}, G_{max}]$ for some maximum gradient value G_{max}. k is often chosen equal to 256 [**4**].

2.3. Discretisation. Of course, when we record a MRI signal, this happens in discrete time intervals. As such, we apply discrete Fourier Transform algorithms such as FFT. These are thoroughly investigated in [**1**, Ch. 5]. Care must be taken that the number of sampling points is large enough; if we encode the frequencies in the range of $\omega_N - \omega_0$ and record a signal for T seconds, we will need at least $T(\omega_N - \omega_0)$ sampling points. If we take less sampling points, aliasing will occur; in the final image will contain copies of only a part of the object we wanted to visualise [**3**, Ch. 5] [**1**, Ch. 5].

If we look at the frequencies present in the frequency encoding direction, we see that the spins will rotate at different frequencies. Assumr that the frequency gradient is centralised at the origin, i.e., that the magnetic field strength at position x is given by equation (6) [**2**]. This gives us that the minimum frequency is at $x_{min} < 0$ and the maximum frequency is at $0 > x_{max}$ while $x_{max} = -x_{min}$ and $x_{max} - x_{min} = \Delta x$. Thus the sampling rate N/T should equal:

$$\frac{N}{T} = \gamma(B_0 + x_{max}G_x - B_0 - x_{min}G_x) = \gamma G_x \Delta x \tag{19}$$

we can rewrite this to:

$$\Delta x = \frac{N/T}{\gamma G_x} \tag{20}$$

in this last formula, we clearly see that if we take the sampling rate $\frac{N}{T}$ too low, Δx has to get lower as well. And if Δx is smaller than the object size, aliasing (or wrap-around) will occur, as we have seen. Δx is thus also called the *field of view (FOV)*. For the y-direction, or the phase encoding direction, we obtain a similar relation. Since we have seen that in the 2D FT transform the phases are seen as frequencies by using many different phase encoding gradients G_y, we have to identify frequencies in between ξ_{min} and ξ_{max}. It is not directly clear how to calculate these minimum and maximum phase shifts. However, it turns out [**3**, Ch. 7] that the following relation holds:

$$\int_0^t G_{y,max} dt = \frac{S}{2\gamma\Delta x} \tag{21}$$

where t is the amount of time the phase encoding gradient is turned on, $G_{y,max}$ is the maximum phase encoding gradient, and S is the number of phase encoding steps applied.

2.4. k-Space. Consider a k x l matrix S with entries $S_{i,j}$, $0 \leq i \leq k$, $0 \leq j \leq l$. Let $\phi_{G_{y,i}}(x)$ be the function denoting the frequency at position x, and let $\xi(y)$ be the phase shift at position y. We subdivide the xy-plane into k^2 squares of equal size. Thus if we let the centre of our xy-plane be denoted by $(0,0)$, and the size of our slice be given by Δx and Δy, we obtain k^2 squares of size $\Delta x/k$ by $\Delta y/k$. The nuclei at each square will have their own net magnetisation vectors; each square denotes a packet. The signal we receive by NMR at a given phase gradient G_y is the summation of the transversal component of each of those vectors.

Let the signal be denoted by $f(G_{y,i}, t)$. Suppose we sample the signal at l equal time intervals and that we store those values at $S_{i,j}$. If we repeat this procedure k times for each matrix row i, we obtain the so-called k-Space given by the matrix S. Now applying a discrete FT algorithm such as FFT in one direction and then the other, will yield a k by l image constructed out of the raw data [**2**].

2.5. Quadrature detection. Suppose we have a spin packet with a vector in an xy-plane rotating clockwise starting at the point $(0, 1)$. This net vector will make a full circle around the origin and return back at the point $(0, 1)$ after some time T; which is the Larmor frequency. Suppose we have a coil which we use to read out the signal somewhere on the sides of the xy-plane; let it be placed alongside the y-axis. At time 0, the vector points to $(0, 1)$; its coordinate on the y-axis is 1 at time 0: $y(0) = 1$. Since the spin vector makes a circle with period T, we know now that $y(t) = cos(2\pi Tt)$.

Now consider the function $\tilde{y}(t) = \cos(-2\pi Tt)$. This function describes a vector rotating *counter*-clockwise starting at $(0, 1)$. Of course, since the cosine is even, we have that $\tilde{y} = y$. Hence when we only look at the information along the x-axis, we do not obtain information about the direction of spin; applying an FT will yield nonzero results at period T as well as $-T$ on both functions y and $\tilde{y}$. This can be prevented by applying *quadrature detection*.

This works by not only placing reading out function components along the y-axis, but also the x-axis; we thus place another coil along the x-axis. If the vector rotates clockwise, we obtain for the x-components of that value the function $x(t) = sin(2\pi Tt)$ and for counter-clockwise rotating vectors $\tilde{x}(t) = sin(-2\pi Tt) \neq x(t)$. Thus we see that the pair $(x(t), y(t))$ identifies a clockwise turning vector, and $(x(t), \tilde{y}(t))$ a counter-clockwise turning one.

Likewise, if we input the FT information not only concerning the x-axis, but also information about the y-axis, the FT is able to detect the rotation direction of spin vectors. We can do this by giving input about the x-components as real data, and the y-components as imaginary data. Passing only either x or y data to the FT is called *linear detection* and has been used in older equipment [**3**, Ch. 5].

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APPENDIX A

The spin-echo sequence

The time required before a net magnetisation vector $N = (N_x, N_y, N_z)$ after a 90 degrees pulse is relaxed back in to its original equilibrium state, is measured by equation (5):

$$M = M_{tra} = M_0 e^{-\frac{t}{T_2^*}} \tag{22}$$

where M denotes the length of the transversal component of the net magnetisation vector; $M = ||(N_x, N_y)||$. In equilibrium, $M = 0$ and $N_z = M_0$, while directly after a 90 degrees pulse, $M = M_0$ and $N_z = 0$.

The decay of M is, in the first place, because of the tendency to return to the equilibrium state. This is caused by molecular interactions. The time required to fully relax N differs for different kind of material. However, since we are unable to produce perfectly homogeneous magnetic fields, the process of transversal relaxation is accelerated. In general, we have that:

$$1/T_2^* = 1/T_2 + 1/T_2^i$$

where T_2 is the *pure* T_2, i.e., unaffected by magnetic disturbances, and T_2^i is the factor bringing into account those disturbances [**3**, Ch. 3].

180 degrees pulse. To weed out the effect of the magnetic disturbances, we may apply the so called *spin-echo* sequence. In order to describe this sequence, we first introduce a second RF pulse similar to the 90 degrees pulse: the 180 degrees pulse. This pulse drives the net magnetisation vector to $M = 0$ and $N_z = -N_0$ (a 180 degree rotation); it puts all nuclei which it affects into the high energy state.

Suppose we first apply a 90 degrees pulse. We then have that N is pushed into the xy-plane. After the pulse, de-phasing begins and M gradually decreases until we are back in the equilibrium state. However, what happens if we apply a 180 degrees pulse during relaxation? Since N is rotated 180 degrees, all its three dimensional components simply invert; i.e., the net magnetisation vector after the pulse becomes $N_{180} = (-N_x, -N_y, -N_z)$. Since the spin direction did not change, the relaxation occurring after the 180 degrees pulse actually first puts N back in to the xy-plane as $-N_z$ tends to 0, after which relaxation again proceeds as usual until $M = 0$ and $N_z = N_0$.

Spin-echo. If we record a signal during this sequence, we will first see a normal signal appear as we saw in the first section. Then somewhere after the peak of this original function, we will see that M increases as the vector relaxes back into the xy-plane after a 180 degrees pulse. After reaching a peak when $N_z = 0$, M fades to 0 in the usual manner. The part of the signal recorded after the 180 degrees pulse is called the *echo signal* of the original signal.

The peaks of the original signal and the echo are not of the same intensity; as time proceeded, $|N_z|$ was still steadily increasing while $|M|$ was decreasing as in (22). The 180 degrees pulse did thus have no influence on the transversal relaxation caused by the molecular interactions. The difference in peak heights can then be only due to T_2 since the magnetic disturbances on M have been nullified by the 180 degrees pulse when $N_z = 0$ [**2**].

Hence this sequence can be used to find the pure T_2 of the object we image. For greater accuracy, one may choose to apply multiple 180 degrees pulses each time the net magnetisation vector is relaxing above the xy-plane. This will yield a better T_2 estimate. This sequence is called spin echo. The time between the two peaks of the original and the echo is called TE. The time between the 90 degrees and the 180 degrees pulses then is $TE/2$. 180 degrees pulses are repeated after TE seconds. After TR seconds, the whole sequence (from the 90 degrees pulse and further) is repeated. TR and TE are the values which can be manually set during MRI experiments and the quality of the image is heavily dependent on these settings, which typically differ for different materials [**2**].

T1 and T2 weighted images. Remember that the properties of material include T_1, and T_2 from equation (18). Hence if we image a part of the body containing different tissues, for example, those different tissues will have different $T_{1/2}$ and thus differ on the resulting MRI image, *when TE and TR are chosen well enough.* Because if TE and TR are much larger than the T_2 and T_1 respectively, then both e^{-TR/T_1} and e^{-TE/T_2} will go to zero; the T1 and T2 processes will not have any influence on the intensities at the pixels of the resulting MRI images.

On the other hand, if we choose TR small, the time before the pulse sequences are repeated may be smaller than the time required to complete the T1 process. Thus the image will be more dependent on T1-properties of tissues. On the other hand, if we also choose $T1$ to be short, there is not much time for $T2$ relaxation; choosing it small enough causes the resulting image to be almost independent of T_2 properties of tissues. Hence with short TE and TR, the resulting MRI image is T1-weighted[**2**].

If we want to obtain a T2-weighted image, we have to let TR become quite large so that all tissues have completed their T1 processes. If we then choose the TE long enough to let the T2 process happen, but not too long so that all T2 processes at each tissue is already done, we obtain the asked T2 weighted image [**2**]; tissues with a large T_2 will contribute longer to the final MRI image.